Exact distributions of eigenvalue curvatures for time-reversal-invariant chaotic systems

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The parametric motion of eigenvalues of chaotic quantum systems is studied by means of the distribution of eigenvalue curvatures k (second derivative with respect to a perturbation parameter). Using supersymmetric integral representations, this distribution is computed *exactly* for the orthogonal ($\beta = 1$) and the symplectic ($\beta = 4$) ensemble. It is found that $P(k) \propto [1 + k^2]^{-(2+\beta)/2}$ in agreement with a recent conjecture by Zakrzewski and Delande [Phys. Rev. E **47**, 1650 (1993)].

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Both theoretically and experimentally, one frequently studies the dependence of the energy levels on external parameters such as magnetic and electric fields. For a detailed description of the parameter dependences one needs to solve explicitly the Schrödinger equation for the system. However, at times this may not be feasible or even desired. In fact, for quantum systems with chaotic classical analogs one often focuses on the statistical properties of their energy spectra [1]. It has been known for some time that these exhibit universal behavior in the absence of external parameters, depending only on the fundamental symmetries of the Hamiltonian, and that they can be described by random-matrix theory [1–3]. There exist three universality classes: Systems without time-reversal invariance fall into the unitary ensemble. Time-reversal-invariant systems belong to the orthogonal or symplectic ensemble depending on the absence or the presence of strong spin-orbit scattering, respectively.

More recently, it has become clear that also the parametric dependence of eigenvalues of complex quantum systems exhibits universal behavior [4–7]. Mainly two types of quantities characterizing the parametric motion of eigenvalues have been studied. First, one can consider correlators of the density of states at different values of the external parameter [6–10]. Such correlators arise, for example, in studying parametric correlations of thermodynamic properties of mesoscopic systems [11]. Second, one can also study the statistics of the derivatives of the eigenvalues $E_n(\lambda)$ with respect to a perturbation parameter λ [5,12–18]. In this paper I consider the distribution of the eigenvalues curvatures K_n, which are defined as the second derivative of the eigenvalues,

$$K_n = \frac{d^2 E_n(\lambda)}{d\lambda^2}. \tag{1}$$

The curvature distribution $P(K)$ was introduced by Gaspard *et al.* [5], who showed that its asymptotic behavior is universal for large curvatures, $P(K) \sim K^{-(2+\beta)}$, where $\beta = 1, 2$, or 4 for the orthogonal, unitary, and symplectic ensembles, respectively. A simple argument for this result goes as follows [13]. Typically, the curvature is large close to avoided level crossings where one effectively has a two-level system. Anticipating Eq. (6) below, one has $K \sim 1/S$ for two-level systems, where S denotes the level spacing. Changing variables in the well-known spacing distribution $P(S) \sim S^\beta$ for small S, one immediately recovers the asymptotic behavior of $P(K)$ mentioned above. Subsequently, this result was verified numerically for a wide variety of systems [12–14,19]. Moreover, Zakrzewski and Delande [14] found that for the kicked top the full distribution could be fit quite well to very simple functions

$$P(k) = C_\beta[1 + k^2]^{-(2+\beta)/2}, \tag{2}$$

where C_β is the normalization constant. The distribution is expressed in terms of the dimensionless curvature

$$k = \frac{K\Delta}{\beta\pi\langle(dE_n/d\lambda)^2\rangle}, \tag{3}$$

where Δ denotes the average level spacing. Previously, I proved that this is indeed the exact distribution for the unitary ensemble [17]. It is the purpose of the present paper to derive the *exact* curvature distributions for the orthogonal and symplectic ensembles. Again, I find that the conjecture (2) due to Zakrzewski and Delande [14] is in fact the exact result.

The basic idea for the calculation is similar to that for the unitary ensemble. The curvature distribution is related to averages over determinants of random matrices. However, while for the unitary ensemble this average could be represented in terms of an integral over a coset of the ordinary group SU(4) [17], one encounters integrals over cosets of supergroups involving both commuting and anticommuting variables in the calculations for the orthogonal and symplectic ensembles. In contrast to most supersymmetry calculations, these supergroups have different numbers of bosonic and fermionic dimensions.

The parametric dependence of energy levels can be included in random-matrix theory by starting from a one-parameter family of Hamiltonians

$$H(\lambda) = (\cos\lambda)H_1 + (\sin\lambda)H_2, \tag{4}$$

where both H_1 and H_2 are random $N\times N$ ($2N\times 2N$) matrices drawn from the Gaussian orthogonal (symplectic) ensemble with probability distribution

$$P(H_1, H_2) \sim \exp\left\{-\frac{1}{2}N\mathrm{tr}(H_1^2 + H_2^2)\right\}. \tag{5}$$

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In the orthogonal ensemble the Hamiltonian is a real symmetric matrix, while in the symplectic ensemble it satisfies the time-reversal relation $H^* = (-i\sigma_y)H(i\sigma_y)$, where the Pauli matrix σ_y acts on the spin indices of the Hamiltonian. Note that each eigenvalue is doubly degenerate in the symplectic ensemble due to the Kramers degeneracy. With these definitions the curvature distribution is independent of λ and hence only $\lambda = 0$ is considered in the following. It can also be shown that quite generally it is sufficient to average over the unperturbed Hamiltonian H_1 [20]. However, the calculation is greatly simplified when averaging over both H_1 and H_2. Hence I take this approach in the present paper.

Second-order perturbation theory yields an *exact* expression for the eigenvalue curvatures

$$K_n = -E_n + 2 \sum_{m\,(\neq n)} \frac{|(H_2)_{nm}|^2}{E_n - E_m}. \tag{6}$$

Here the E_n denote the eigenvalues of H_1. The curvature distribution is defined as [17]

$$P(K) = \frac{1}{\rho(0)} \left\langle \sum_n \delta(E_n)\delta(K - K_n) \right\rangle_{H_1,H_2}. \tag{7}$$

To eliminate density-of-states effects only levels at the center of the semicircle spectrum [2] (i.e., $E_n = 0$) are included. The distribution is normalized with the average density of states $\rho(0)$ at $E = 0$.

I start by performing the average over the perturbation H_2. Upon using the Fourier representation for the second δ function in Eq. (7) this average becomes a Gaussian integral, which yields

$$P_\beta(K) \sim \int_{-\infty}^{\infty} d\alpha\, e^{iK\alpha} \times \left\langle \delta(E_1) \prod_{n=2}^{N} \left(\frac{E_n}{E_n - i2\alpha/N} \right)^{\beta/2} \right\rangle. \tag{8}$$

Note that overall prefactors are consistently dropped since they can be reconstructed from the normalization condition at the very end of the calculation. Here the fact was used that, statistically, the curvatures of all levels are equivalent allowing one to consider only, say, K_1. For the unitary ensemble it is possible to perform the α integration at this stage [17]. This is not possible or helpful in the orthogonal and symplectic ensembles where instead one must compute directly the average on the right-hand side of (8) using the well-known joint-eigenvalue distribution [2]

$$P_\beta(E_n) \sim \prod_{i<j} |E_i - E_j|^\beta \exp\left\{ -\frac{1}{2} N \sum_{j=1}^{N} E_j^2 \right\}. \tag{9}$$

Treating the eigenvalue E_1 separately, one sees that the joint-eigenvalue distribution factorizes into the joint-eigenvalue distribution for an $(N-1)$-dimensional random matrix H with eigenvalues $E_2, \ldots, E_N$, a term $|\det(E_1 - H)|^\beta$, and a function of E_1. Hence one can express the Fourier transform $P(\alpha)$ of the curvature distribution in terms of an average over the $(N-1)$-dimensional random matrix H. For the orthogonal ensemble one obtains

$$P_{\beta=1}(\alpha) \sim \left\langle \frac{|\det H|\sqrt{\det H}}{\sqrt{\det[H - i2\alpha/N(N-1)]}} \right\rangle_H^{(N-1)}. \tag{10}$$

In the limit of large random matrices, the average may be taken over an N-dimensional random-matrix ensemble. Hence I drop the superscript $(N-1)$ on the average in the following. Writing $|\det H| = (\det H)^2/(\sqrt{\det H})^*\sqrt{\det H}$ [21], one finds

$$P_{\beta=1}(\alpha) \sim \frac{(\det H)^2}{(\sqrt{\det H})^*\sqrt{\det(H - i2\alpha/N)}}. \tag{11}$$

Since $P_\beta(\alpha) = P_\beta(-\alpha)$, I restrict myself to $\alpha < 0$ in the following. The determinants on the right-hand side of Eq. (11) can be represented in terms of Gaussian integrals over bosonic (commuting) and fermionic (anticommuting) variables. To this end, introduce a supervector $\phi^T = [S^1, S^2, \chi^1, (\chi^1)^*, \chi^2, (\chi^2)^*]$ with real bosonic entries S^i and complex fermionic entries χ^i, each entry being itself an N-dimensional vector. One has

$$P_{\beta=1}(\alpha) \sim \int [d\phi] \exp\left\{ \frac{i}{2}\phi^\dagger L^{1/2} H L^{1/2}\phi + \frac{\alpha}{N}\phi^\dagger P \phi \right\}, \tag{12}$$

where the measure is $[d\phi] = \prod_{j=1}^{N}\prod_{i=1}^{2} dS_j^i d\chi_j^i (d\chi_j^i)^*$, the matrix $L = \mathrm{diag}[1,-1,1,1,1,1]$, and the projector $P = \mathrm{diag}[1,0,0,0,0,0]$. The Hamiltonian H stands for the direct product of the true $N \times N$ Hamiltonian and the unit matrix in the superspace. Averaging over the Hamiltonian and performing the Hubbard-Stratonovich transformation in the usual manner [22], one finds an integral representation

$$P_{\beta=1}(\alpha) \sim \int [d\sigma] \exp\{-(N/8)\mathrm{trg}\sigma^2 - (N/2)\mathrm{trg}\ln(\sigma - 4i\alpha P/N)\}, \tag{13}$$

where trg denotes the graded trace. The supermatrix σ has two bosonic and four fermionic dimensions and can be parametrized as $\sigma = T\Lambda T^\dagger$ with Λ a diagonal matrix and T an element of the graded Lie group UOSP(1, 1|4). More explicitly, T is a pseudounitary matrix satisfying $T^\dagger L T = L$ with additional structure $T^T C T = C$ due to time-reversal invariance where $C = \mathrm{diag}[\sigma_z, -i\sigma_y, -i\sigma_y]$. In the limit of large random matrices $(N \to \infty)$ the σ integration can be performed by the saddle-point method. One may also expand the exponent in (13) to first order in α/N. Deriving the saddle-point equations, one finds that the relevant saddle point is $\Lambda_0 = i\sqrt{2}\,\mathrm{diag}[1,1,1,1,-1,-1]$. The signs of the entries in the boson-boson block are fixed by convergence requirements, while the choice of signs in the fermion-fermion block corresponds to the largest saddle-point manifold. Due to the presence of the projector P in the exponent, only the first row of T appears in the integrand and hence the integration can be restricted to the coset manifold UOSP(1,1|4)/UOSP(1|4). The first row of T can be parametrized as $[x, y, \eta, \eta^*, \zeta, \zeta^*]$ with the constraint $x^2 - y^2 + 2\eta^*\eta + 2\zeta^*\zeta = 1$ due to pseudounitarity (x,y are real bosonic variables, η,ζ are complex fermionic). It can be shown and it is intuitively plausible that the invariant measure on this manifold is

$$d\mu(T) = dx\, dy\, d\eta\, d\eta^*\, d\zeta\, d\zeta^* \times\delta(1 - x^2 + y^2 - 2\eta^*\eta - 2\zeta^*\zeta). \qquad (14)$$

Now, the Fourier transform of the curvature distribution becomes

$$P_{\beta=1}(\alpha) \sim \int d\mu(T) \exp\{\sqrt{2}\alpha(x^2 + y^2 + 2\eta^*\eta - 2\zeta^*\zeta)\}. \qquad (15)$$

Performing this integral (for example, by employing the Fourier representation of the δ function in the measure) and doing the Fourier transform, one finds for the curvature distribution $P_{\beta=1}(K) \sim [2 + K^2]^{-3/2}$. Computing the dimensionless curvature $k = K/\sqrt{2}$, one obtains the final result

$$P(k) = \frac{1}{2}[1 + k^2]^{-3/2}, \qquad (16)$$

which proves the Zakrzewski-Delande conjecture for the orthogonal ensemble.

In the symplectic ensemble, one finds from Eqs. (8) and (9) that

$$P_{\beta=4}(\alpha) \sim \left\langle \frac{(\det H)^3}{\det(H - i\alpha/N)} \right\rangle_H . \qquad (17)$$

To represent the determinants by bosonic and fermionic Gaussian integrals, introduce a supervector

$$\phi^T = [S_\uparrow, S_\downarrow, S_\downarrow^*, -S_\uparrow^*, \chi_\uparrow, \chi_\downarrow, \chi_\downarrow^*, -\chi_\uparrow^*, \eta_\uparrow, \eta_\downarrow, \eta_\downarrow^*, -\eta_\uparrow^*, \zeta_\uparrow, \zeta_\downarrow, \zeta_\downarrow^*, -\zeta_\uparrow^*], \qquad (18)$$

where each entry is itself an N-dimensional vector. Bosonic components are denoted by S, fermionic components by χ, η, ζ. For $\alpha < 0$ one finds

$$P_{\beta=4}(\alpha) \sim \int [d\phi] \left\langle \exp\left\{ \frac{i}{2}\phi^\dagger (H - i\alpha P/N)\, \phi \right\} \right\rangle_H . \qquad (19)$$

Here, the projector $P = \text{diag}[1, 1, 0, 0, 0, 0, 0, 0]$. This leads to an integral representation

$$P_{\beta=4}(\alpha) \sim \int [d\sigma] \exp\{-(N/2)\text{trg}\sigma^2 - N\text{trg}\ln(\sigma - i\alpha P/N)\}. \qquad (20)$$

The supermatrix σ has two bosonic and six fermionic dimensions and can be represented as $\sigma = T\Lambda T^\dagger$ with Λ a diagonal matrix and T a unitary matrix, $T^\dagger T = 1$. In addition, T satisfies the relation $T^T C T = C$, where $C = \text{diag}[i\sigma_y, 1, 1, 1]$, due to time-reversal symmetry. Hence T is an element of UOSP(6|2). One finds that the relevant saddle point is $\Lambda_0 = i\text{diag}[1, 1, 1, 1, 1, 1, -1, -1]$. In the orthogonal ensemble I restricted the integration over T to the coset manifold defined by the projector P. In the present case, this leads to problems with boundary terms and hence I choose to integrate over the saddle-point manifold UOSP(6|2)/UOSP(2|0)⊗UOSP(4|2) defined by Λ_0,

$$P_{\beta=4}(\alpha) \sim \int d\mu(T) \exp\{-i\alpha\text{trg}T\Lambda_0 T^\dagger P\}, \qquad (21)$$

where $d\mu(T)$ is the invariant measure on the saddle-point manifold. An element T of the saddle-point manifold can be written in the block form

$$T = \begin{bmatrix} \sqrt{1 - tt^\dagger} & t \\ -t^\dagger & \sqrt{1 - t^\dagger t} \end{bmatrix}, \qquad (22)$$

where t is a 6×2 matrix. The first two rows of t have fermionic entries while the remaining rows are bosonic. From the time-reversal condition for T it follows that $t^* = Dt$ with $D = \text{diag}[i\sigma_y, 1, 1]$. In particular, this implies that t involves two independent complex fermionic variables. Expressing the integral over the saddle-point manifold in terms of t, one has

$$P_{\beta=4}(\alpha) \sim \exp\{2\alpha\} \times \int d\mu(T) \exp\{-2\alpha[(tt^\dagger)_{11} + (tt^\dagger)_{22}]\}. \qquad (23)$$

Only the two fermionic elements of t enter into the integrand. This observation can be used to prove the Zakrzewski-Delande conjecture for the symplectic ensemble without explicit evaluation of the integral. Equation (23) implies that $P_{\beta=4}(\alpha) = p_2(\alpha)\exp\{2\alpha\}$, where $p_2(\alpha)$ is a polynomial of second order in α. Fourier transforming and demanding that asymptotically $P_{\beta=4}(K) \sim 1/K^6$ (a result which was proven by Gaspard *et al.* [5]), one finds that this determines $P_{\beta=4}(K)$ uniquely,

$$P(k) = \frac{8}{3\pi}[1 + k^2]^{-3}. \qquad (24)$$

Here I used $k = K/2$. This completes the prove of the Zakrzewski-Delande conjecture for all three ensembles.

It is worthwhile to note that the distribution (2) for the unitary ensemble can also be obtained from the approach taken in this paper. One finds from Eqs. (8) and (9) that

$$P_{\beta=2}(\alpha) \sim \left\langle \frac{(\det H)^3}{\det(H - i\alpha/N)} \right\rangle_H . \qquad (25)$$

The corresponding supersymmetric integral representation involves $\sigma = T\Lambda T^\dagger$ with $T \in$ U(1|3). The relevant saddle point has the signature $\Lambda_0 \sim \text{diag}[1, 1, 1, -1]$. Integrating over the saddle-point manifold one recovers the result (2) previously proven in Ref. [17].

In summary, I have computed the exact distributions of eigenvalue curvatures for the orthogonal and symplectic ensembles of random-matrix theory. The results are in agreement with a recent conjecture by Zakrzewski and Delande [14] based on a careful study of numerical results. It is quite remarkable that the exact curvature distribution has such a simple functional form. This is in marked contrast to other distribution functions in random-matrix theory such as the level-spacing distribution [2]. It would be interesting to uncover the deeper reason for this simplicity. A possible line of thought may start from the observation that the expression for the eigenvalue curvature (6) involves a sum over a large number of (statistically dependent) terms [23]. This suggests that the simplicity of the curvature distribution may be due to some generalized central-limit theorem.

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